01A_Elastic Constants

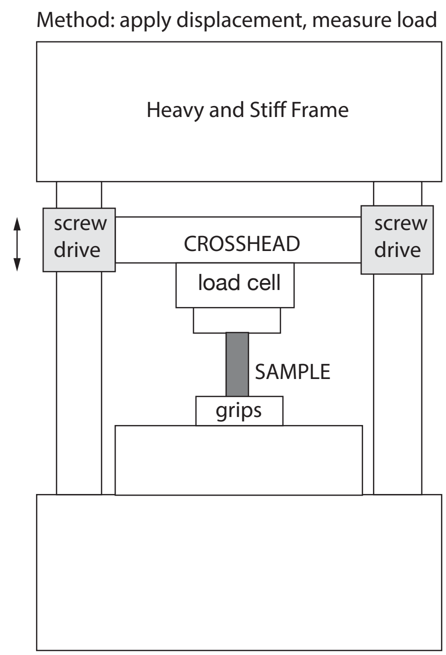
Topics:

1. Elastic Constants

2.The need for at least two elastic constants

3. Principal stresses and strains for finding relationships between E, ν and G

The Instron (for uniaxial tensile test)



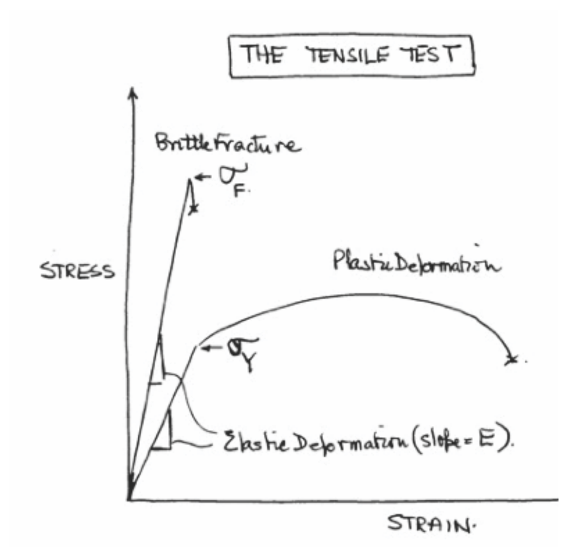


Fig. 2

HW 01A.1
Explain why such a heavy machine is used to test samples that are much smaller.

Remember that the Instron measures the load in response to the application of a displacement, which is applied by the screw drive. Therefore, displacement is the independent variable. The (experimental) load displacement graphs are converted into stress-strain graph by dividing load by the cross-section area and the displacement by the gage length of the sample.

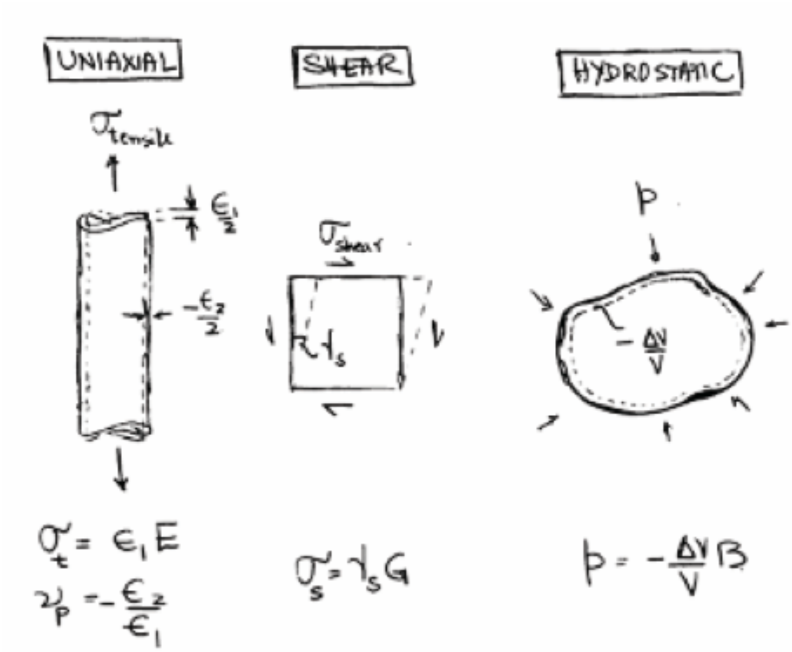
$$\sigma = \frac{P}{A}, \quad \varepsilon = \frac{u}{L_o}$$

These equations are valid for elastic deformation only where the strains are nearly always <1% (see large strain behavior under "Global").

Note: the load displacement curves are valid for a specimen of a specific geometry. However, stress and strain are global variables which lead to fundamental material parameters such as the Youngs Modulus.

Youngs Modulus (E), Shear Modulus (G), Bulk Modulus (B) and Poissons Ratio (ν)

Experiments to measure elastic properties of a material are measured in fundamentally different ways:

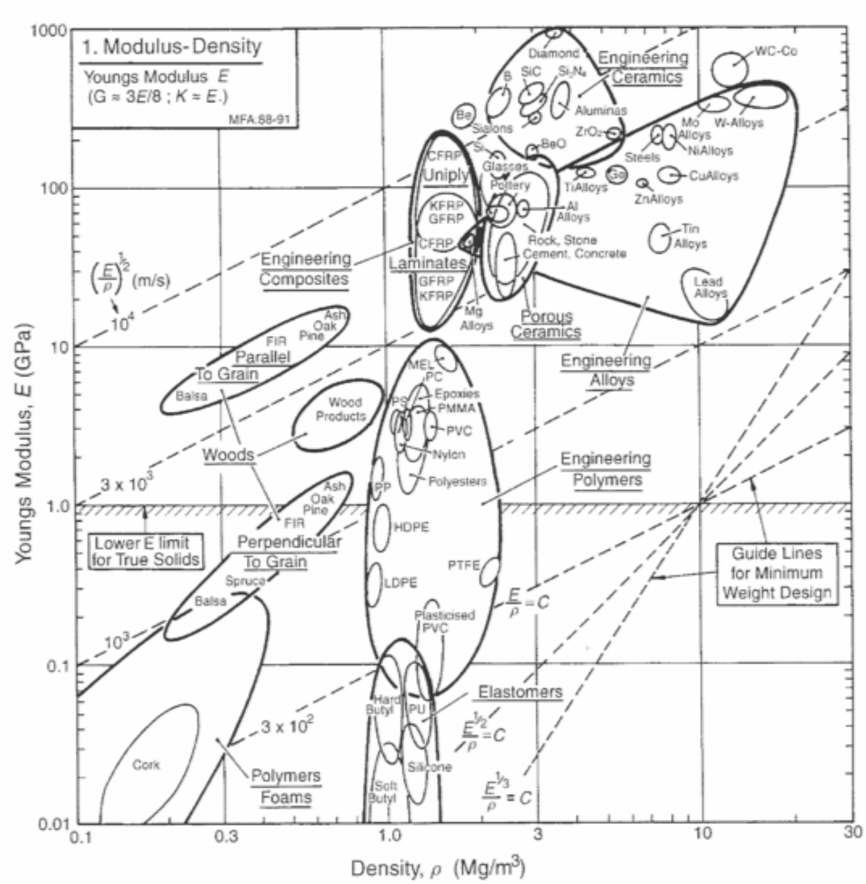


The Instron test is an example of uniaxial deformation.

Torsion of a cylindrical specimen an example of shear deformation.

Deformation under hydrostatic pressure in cells, often used by geologists are examples of hydrostatic deformation. Typical pressure in these cells can range from 1 kbar to 10s of kbars.

Remember 1 bar = 100 kPa; therefore 1 kbar = 100 MPa.



How many elastic constants?

The figure above shows four of them:

The Youngs Modulus, E (GPa)

The Shear Modulus, G (GPa)

The Bulk Modulus, B (GPa)

The Poisson's Ratio ν (dimensionless)

The first three conform to the following description

stress = (elastic constant) x (strain)

HW 01A.2: (i) Create a hierarchy of different materials classes in the order of increasing elastic modulus using the information given in the map on the right. (ii) Composites made from polymers and carbon fibers have their modulus closer to that of carbon. Can you model the modulus of the composite made with 70/30 carbon/polymer mixture (by vol). (iii) Which material would you use for making a diving board - explain why?

How many independent elastic constants for an "isotropic" material?

The following observations are possible from the figures for the three types of deformation:

1. *Uniaxial deformation:* It is clear that at least two elastic constants are needed, one to describe the stress-strain behavior in the uniaxial direction, and the other for the strain in the transverse direction when strain in applied longitudinally. The relationship between transverse and axial strain will vary from one material to another, and therefore needs its own, independent description. It is defined as

   $$\nu = -\frac{\varepsilon_2}{\varepsilon_1} \qquad (01A.1)$$

   The Youngs modulus is simple the ratio of the tensile stress and the tensile strain defined as follows

   $$\sigma_1 = E\varepsilon_1 \qquad (0A1.2)$$

   Note that the subscript "1" means deformation along the axis "1" in Cartesian coordinates.

2. *Volume change versus shape change:* It is intuitively obvious that these two fundamental modes of deformation require two different elastic constants. The shear modulus and the bulk modulus represent these two extreme modes of deformation and are defined in that way (noting that pure shear deformation represents a change in shape without a change in volume):

   $$\sigma_s = \gamma_S G \qquad (01A.2)$$

   where γ_S is the shear strain. It is the engineering shear strain, and equal to twice the pure shear strain, called ε_{12} as explained later.

3. *The Bulk Modulus:* This elastic constant relates the volume change in response to an applied hydrostatic pressure

   $$p = -B\frac{\Delta V}{V_o} \qquad (01.A.3)$$

   Note the sign convention. Pressure is compressive stress and therefore a negative quantity while $\frac{\Delta V}{V_o}$ is a positive quantity, and B the bulk modulus is a material property and is therefore a scalar without sign.

The sign convention for stress and strain is as follows: these quantities are positive if they refer to tensile and negative if to compressive deformation. This is a universal rule.

The following table summarizes how different modes of deformation (uniaxial, shear and hydrostatic) represent volume and shape changes:

Type of Deformation	Shape Change	Volume Change
UNIAXIAL	yes	yes
SHEAR	yes	no
HYDROSTATIC	no	yes

Since we have inferred that two independent elastic constants are needed, and no more, to describe the full elastic behavior, it is implied that any two of the four elastic constants, described above, can be chosen to be independent. The remaining two can then be described in terms of these two independent constants. For example,

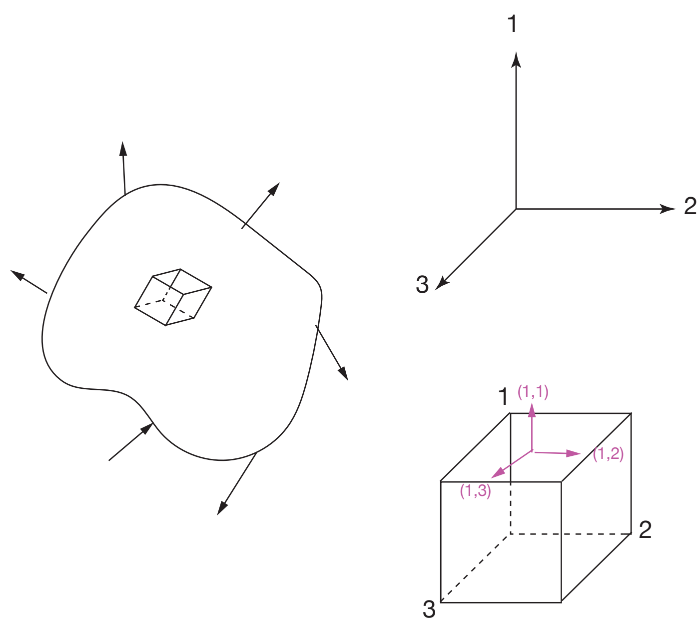
$$G = \frac{E}{2(1+\nu)};\ B = \frac{2G(1+\nu)}{3(1-2\nu)};\ B = \frac{E}{3(1-2\nu)} \qquad (01A.4)$$

Among the three relationships given above the first one that relates the shear modulus to E and ν is most useful.

HW 01A.3: Describe an experiment with an Instron to determine the shear modulus of a material.
HW 01A.4: Show that there are only two indpendent variables in Equations given in (01A.4)

Stress and Strain as a Tensor in Cartesian Coordinates

You can perhaps sense that stress and strain are vectorial quantities. In fact they depend on two directions (a vector has only one). For general description we cut a cube out of a body and consider the local forces on this cube when within the large body as shown below



The mechanics problem is posed by applying forces to the surface of the body such that the forces are in mechanical equilibrium. These forces are called tractions and have the same units as stress, but they are not stresses. They can be normal tractions, i. e. normal to the surface or shear tractions, i. e. tangential to the surface. Now we wish to know the stresses and strain within the body; we pull out a cube from within as shown on the right and describe the stresses and strains in this cube. It is immediately obvious that the stress and strain would be associated with two vectors, one representing the plane and the other the direction of the force. As shown above for plane "1" there are three different quantities for stress and strain. Let us consider the stress; it will have three components: $\sigma_{11}, \sigma_{12}, \sigma_{13}$. thus a general form of the stress tensor emerges:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \qquad (01A.5)$$

The strain tensor can be similarly defined by replacing σ_{ij} by ε_{ij}.

Now the physical meaning of σ_{11}, ε_{11} is clear as it the meaning of off diagonal components of the stress tensor. But the shear represented by the off-diagonal components of the strain tensor needs further clarification. Consider the case for the strain tensor of the following form

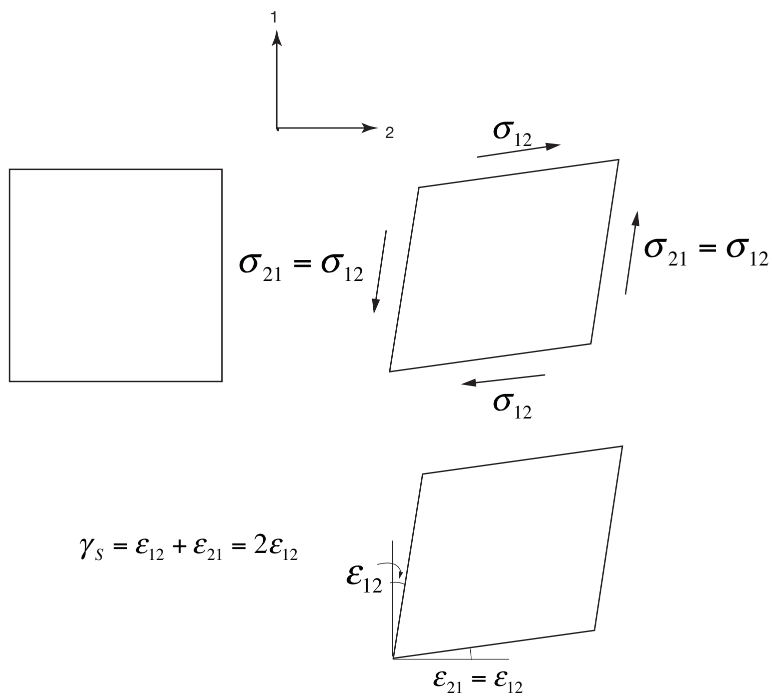
$$\varepsilon = \begin{bmatrix} 0 & \varepsilon_{12} & 0 \\ \varepsilon_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (01A.6)$$

This is the case of pure shear since the diagonal terms are equal to zero.

Two points are of particular note:

1. The shear strain can be expressed in two dimensions (in the 1,2 plane as above), and that the strains are symmetrical.
2. The engineering shear strain, γ_S, used to define the shear modulus is given by $\gamma_S = 2\varepsilon_{12}$.

The pure shear strains are shown in the figure just below



Note again, that the engineering shear strain $\gamma_S = 2\varepsilon_{12}$.

The shear modulus is described in terms of $\sigma_S = \gamma_S G$

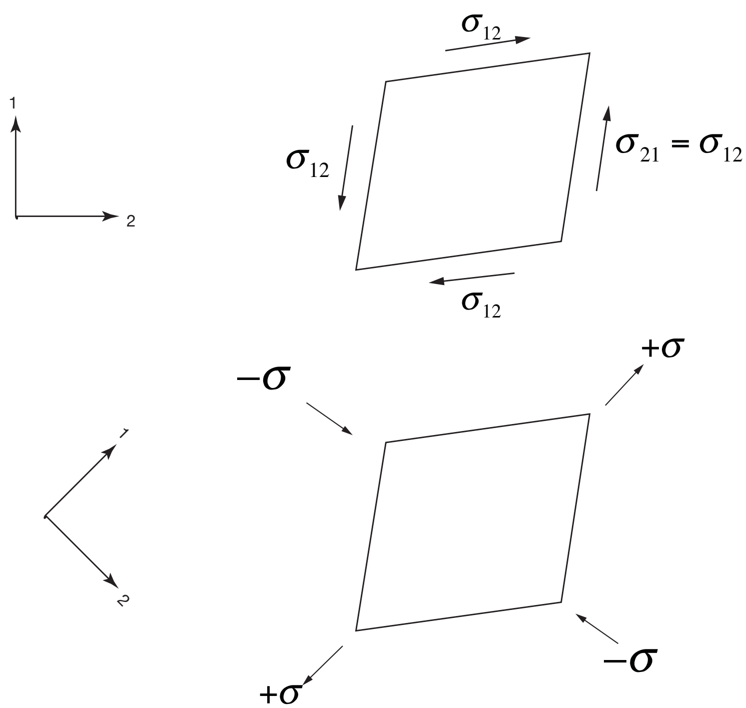
Principal Stresses and Strains

The principal stresses and strains are defined in the particular orientation of the Cartesian axes such that the off-diagonal terms become zero, so that

$$\sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \text{ and } \varepsilon = \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{bmatrix} \qquad (01A.6)$$

Note how the double subscript is no longer needed. Thus the notation is significantly simplified.

The shear in the previous diagram can now be presented in terms of principal stresses and principal strains as follows



The principal stresses and strains for pure shear deformation are

$$\sigma = \begin{bmatrix} +\sigma_1 & & \\ & -\sigma_1 & \\ & & 0 \end{bmatrix} \text{ and } \varepsilon = \begin{bmatrix} +\varepsilon_1 & & \\ & -\varepsilon_1 & \\ & & 0 \end{bmatrix} \qquad (01A.7)$$

Volumetric quantities

The pressure and the volume change can be written in terms of the principal stresses and strains as follows

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$p = -\frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (01A.8)$$

In pure shear deformation, expressed in (01A.8) both quantities in (01A.7) are equal to zero.

Please note that the principal stresses and strains can be used to separate out hydrostatic and shear components of the general stress tensor, as discussed later in this chapter.

Derivation of $B = \frac{E}{3(1-2\nu)}$

Let us consider uniaxial deformation. Here the principal stress and strain tensors are

$$, \sigma = \begin{bmatrix} \sigma_1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \text{ and } \quad \varepsilon = \begin{bmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_2 \end{bmatrix}$$

where

$$E = \frac{\sigma_1}{\varepsilon_1}, \qquad \nu = -\frac{\varepsilon_2}{\varepsilon_1}$$

The definition of the bulk modulus is $-p = B\frac{\Delta V}{V}$

Pressure: $p = -\frac{\sigma_1 + 0 + 0}{3}$

Volume change: $\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_1 + 2\varepsilon_2$

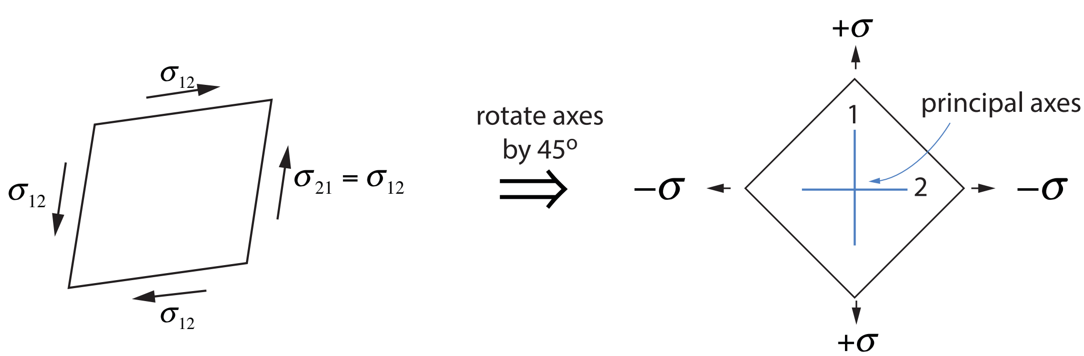
$$\frac{\sigma_1}{3} = B(\varepsilon_1 + 2\varepsilon_2)$$

$$\frac{\sigma_1}{\varepsilon_1} = 3B(1 + 2\frac{\varepsilon_2}{\varepsilon_1})$$

$$B = \frac{E}{3(1-2\nu)}$$

Derivation of $G = \frac{E}{2(1+\nu)}$

The method: consider loading of a specimen in pure shear.



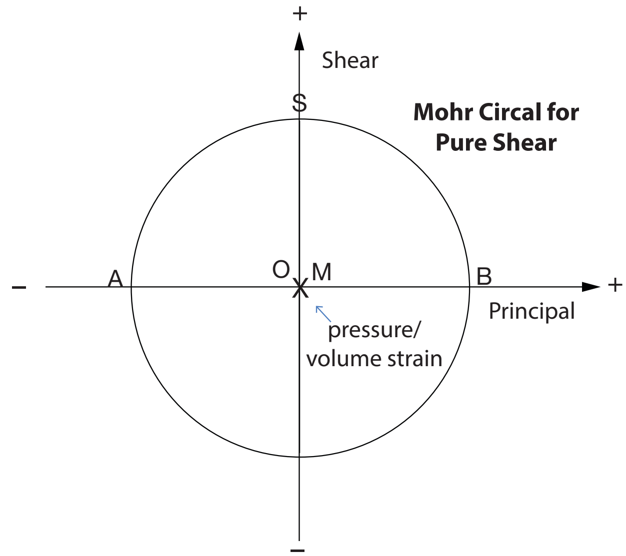
The stress and strain tensors using the principal axes are

$$[\sigma] = \begin{bmatrix} +\sigma & & \\ & -\sigma & \\ & & 0 \end{bmatrix} \text{ and } [\varepsilon] = \begin{bmatrix} \varepsilon & & \\ & -\varepsilon & \\ & & 0 \end{bmatrix} \qquad (01A.9)$$

The strain in the "z" direction (normal to the plane of the paper) is zero since the Poisson strain from $+\sigma$ and $-\sigma$ will cancel out since the first will produce a negative strain and the second a positive strain of the same magnitude.

Recall that shear strain does not involve volume change therefore the pressure and the volumetric strain must be zero. The pressure is equal to the sum of the principal strains divided by 3, while the volumetric strain is equal to the sum of the principal strains.

Now we need to extract the shear stress and the shear strain from Eq. (01A.9). For this purpose, we resort to the Mohr Circle. Its construction for zero pressure and zero volume change is given graphically as shown on the right. The two principal stresses in pure shear are shown by the horizontal axis, and the pure shear by the vertical axis, to that



$$OB = +\sigma$$
$$OA = -\sigma$$
$$M\,(\text{the mean}) = 0(\text{the pressure})$$

Therefore, the shear stress is the radius of the circle

$$\sigma_S = \sigma$$

Applying the same procedure to the strain we get that

$$\varepsilon_S = \varepsilon$$

and as explained before the engineering shear strain is given by

$$\gamma_S = 2\varepsilon$$

Now be definition the shear modulus is given by

$$G = \frac{\sigma_S}{\gamma_S} = \frac{\sigma_S}{2\varepsilon_S} \qquad (01A.9)$$

Now it remains to relate σ_S / ε_S to E and ν. First we express the stress tensor as a sum of two separate tensors

$$\begin{bmatrix} +\sigma & & \\ & -\sigma & \\ & & 0 \end{bmatrix} = \begin{bmatrix} +\sigma & & \\ & 0 & \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & -\sigma & \\ & & 0 \end{bmatrix} \qquad (01A.10)$$

Each of the terms on the right will yield the following strain tensors

$$[\varepsilon] = \begin{bmatrix} \varepsilon & & \\ & -\nu\varepsilon & \\ & & -\nu\varepsilon \end{bmatrix} + \begin{bmatrix} +\nu\varepsilon & & \\ & -\varepsilon & \\ & & +\nu\varepsilon \end{bmatrix} = \begin{bmatrix} (1+\nu)\varepsilon & & \\ & -(1+\nu)\varepsilon & \\ & & 0 \end{bmatrix}$$

where it is recognized that $E = \sigma / \varepsilon$ from uniaxial experiments.

Applying the Mohr Circle to the principal strains in the above equation we obtain that

$$\begin{aligned} \varepsilon_S &= (1+\nu)\varepsilon \\ \gamma_S &= 2\varepsilon_S = 2\varepsilon(1+\nu) \end{aligned} \qquad \text{(i)}$$

where

$$E = \frac{\sigma}{\varepsilon} \qquad \text{(ii)}$$

and

$$G = \frac{\sigma}{\gamma_S} \qquad \text{(iii)}$$

Combining the above equations $$G = \frac{\sigma = E\varepsilon}{\gamma_S = 2\varepsilon(1+\nu)} = \frac{E}{2(1+\nu)} \qquad (*)$$

In General: any three principal quantities can be separated in to hydrostatic and shear components

For example, consider the general case:

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \sigma_1+\sigma_2+\sigma_3 & & \\ & \sigma_1+\sigma_2+\sigma_3 & \\ & & \sigma_1+\sigma_2+\sigma_3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} (\sigma_1-\sigma_2)+(\sigma_1-\sigma_3) & & \\ & (\sigma_2-\sigma_1)+(\sigma_2-\sigma_3) & \\ & & (\sigma_3-\sigma_1)+(\sigma_3-\sigma_2) \end{bmatrix}$$

The first term on the left represents hydrostatic loading while the second is pure shear, since the sum of the three terms is equal to zero which satisfies the condition for pure shear.

The second stress tensor can be further split into three shear components each of which can be represented in the Mohr Circle, as follows

$$\begin{bmatrix} (\sigma_1-\sigma_2) & & \\ & -(\sigma_1-\sigma_2) & \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & (\sigma_2-\sigma_3) & \\ & & -(\sigma_2-\sigma_3) \end{bmatrix} + \begin{bmatrix} (\sigma_1-\sigma_3) & & \\ & 0 & \\ & & -(\sigma_1-\sigma_3) \end{bmatrix}$$

Here the first tensor is shear in the (1,2) plane, and so on.

The same equations apply to a tensor consisting of three principal strains.

HW 01A.5 Apply the above method to separate the hydrostatic and shear components in a simple uniaxial test for both the principal stress and the principal strain tensors. Use these separated tensors to directly derive the equations

$$G = \frac{E}{2(1+\nu)}; \text{ and } \; B = \frac{E}{3(1-2\nu)}$$